Gravitational waves from inspiraling compact binaries: Accuracy of the post-Newtonian waveforms

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The accuracy of the post-Newtonian waveforms, both in "standard" and "Padé" form, is determined by computing their matched-filtering overlap integral with a reference waveform obtained from black-hole perturbation theory.

I. INTRODUCTION

The need for very accurate model waveforms to search and measure gravitational-wave signals from inspiraling compact binaries was recognized less than six years ago [4]. During this time, post-Newtonian (PN) calculations of binary waveforms have been pushed to high order [3,2]. Currently, the waveform is known to 2.5PN order beyond the quadrupole-formula result, and the 3PN waveform appears to be within reach. An ever important question is: at which order in the post-Newtonian expansion can the waveform be considered to be sufficiently accurate? In the absence of an exact representation of the signal (which might eventually be provided by numerical relativity), a definitive answer to this question remains elusive.

An alternative approach to waveform calculations, based of black-hole perturbation theory, was also pushed forward in the last several years [8,10]. While post-Newtonian theory is based on an assumption of slow orbital motion but allows for arbitrary mass ratios, perturbation theory is based on an assumption of small mass ratios but allows for arbitrary velocities. The physical situation is that of a small mass orbiting a massive black hole, and the emission of gravitational waves is calculated by integrating a linear wave equation — the Teukolsky equation — in the black-hole spacetime. In the limit of a vanishing mass ratio, this approach gives an exact representation of the gravitational-wave signal. When combined with a slow-motion approximation, perturbation theory returns an analytic expression for the waves, in terms of a series in powers of the orbital velicity. Currently, the perturbation-theory waveform is known exactly in numerical form, and at 5.5PN order in analytic form [10].

Although the perturbation-theory waveform is of restricted validity, it is still our best guide in trying to determine the accuracy of the post-Newtonian expansion. In this contribution to these proceedings, this issue will be examined in two different contexts. First, supposing that gravitational-wave signals from small-mass-ratio binaries can be picked up by such interferometric detectors as LIGO/VIRGO, I calculate (Sec. 3) the loss in signal-to-noise ratio incurred by using second post-Newtonian waveforms as search templates. My results suggest that the 2PN waveforms will be acceptably effective for searches. Second, I review (Sec. 4) the very interesting proposal by Damour, Iyer, and Sathyaprakash [6] to use Padé approximants to improve the convergence of the post-Newtonian series. My results suggest that a Padé version of the 3PN waveforms (yet to be produced) will make very accurate templates for searches, and possibly also for measurements. In Sec. 2, I present the framework used throughout this contribution.

The work presented here was partially carried out with Serge Droz. Additional details can be found in [7,9].

II. PRELIMINARIES

The output of an idealized gravitational-wave detector is written as $o(t) = n(t) + s(t)$, where $n(t)$ is stationary Gaussian noise and $s(t)$ is the signal. The detector output is analyzed by matched filtering against a bank of templates $h(t; \boldsymbol{\lambda}, \boldsymbol{\theta})$. Here, $\boldsymbol{\lambda}$ collectively denotes the template's *kinematical* parameters (arrival time and initial phase), while $\boldsymbol{\theta}$ collectively represents the *dynamical* parameters (the two masses, assuming that the binary companions are nonrotating). The statistical properties of the detector noise are summarized by its spectral density, denoted $S_n(f)$, where f is the frequency. Throughout we use the useful approximation [5] $S_n(f) = S_0\Theta(f - f_{\rm min})[(f_0/f)^4 + 2 + 2(f/f_0)^2]$, where the parameters S_0, $f_{\rm min}$, and f_0 are set by the detector. For the initial LIGO, $f_{\rm min} = 40$ Hz and $f_0 = 200$ Hz, while $f_{\rm min} = 10$ Hz and $f_0 = 70$ Hz for the advanced LIGO. The value of S_0 is irrelevant for our purposes.

We define the *ambiguity function* $\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ by

$$\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \frac{(s|h)}{\sqrt{(s|s)(h|h)}}, \tag{1}$$

where, for any functions $a(t)$, $b(t)$,

$$(a|b) = 2\int_0^\infty \frac{\tilde{a}^*(f)\tilde{b}(f) + \tilde{a}(f)\tilde{b}^*(f)}{S_n(f)}\, df \tag{2}$$

is an inner product in the Hilbert space of gravitational-wave signals [5]. Here, $\tilde{a}(f)$ is the Fourier transform of $a(t)$, and an asterisk denotes complex conjugation. The ambiguity function measures the Hilbert-space "angle" between the signal and the template. This "angle" varies with the template parameters, and we are interested in maximizing the ambibuity function over these parameters. We construct the *semi-maximized ambiguity function* (SMAF) by maximizing only over the kinematical parameters:

$$\mathrm{SMAF}(\boldsymbol{\theta}) = \max_{\boldsymbol{\lambda}} \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\theta}). \tag{3}$$

The SMAF will be computed in Sec. 4. The fully maximized ambiguity function is Apostolatos' *fitting factor* [1],

$$\mathrm{FF} = \max_{\boldsymbol{\theta}, \boldsymbol{\lambda}} \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\theta}). \tag{4}$$

The fitting factor is equal to ratio of the *actual* signal-to-noise ratio, obtained with an imperfect set of templates, to the signal-to-noise ratio that *would* be obtained if a perfect set of templates were available. Consequently, the loss in event rate due to template imperfection is $1 - \mathrm{FF}^3$. The fitting factor will be computed in Sec. 3.

In the calculations of Secs. 3 and 4, the reference signal will be provided by black-hole perturbation theory, as was discussed in Sec. 1. Keeping only the dominant mode at twice the orbital frequency, the frequency-domain signal can be expressed as [7]

$$\tilde{s}(f) \propto A(v)\, e^{i\Psi(v)}, \tag{5}$$

where $v = (\pi M f)^{1/3}$ is the orbital velocity (M is the total mass of the binary system, and μ will denote the reduced mass); $A(v)$ is the signal's amplitude, and $\Psi(v)$ its phase. When using the stationary-phase approximation to calculate the Fourier transform, the phase is found to be given by

$$\Psi(v) = \frac{5M}{16\mu} \int^{v} \frac{(v^3 - v'^3) Q(v')}{v'^9 P(v')}\, dv'. \tag{6}$$

Here, $P(v) = (dE/dt)/(dE/dt)_{\mathrm{QF}}$ is the rate at which the gravitational waves remove orbital energy from the system, normalized to the quadrupole-formula expression $(dE/dt)_{\mathrm{QF}} = -(32/5)(\mu/M)^2 v^{10}$; and $Q(v) = (dE/dv)/(dE/dv)_{\mathrm{N}} = (1 - 6v^2)/(1 - 3v^2)^{3/2}$ gives the differential relation between orbital energy and orbital velocity, normalized to the Newtonian expression $(dE/dv)_{\mathrm{N}} = -\mu v$. The functions $A(v)$ and $P(v)$ are calculated by numerically integrating the Teukolsky equation for a small mass moving on a circular orbit of the Schwarzschild black hole.

On the other hand, the templates are given by

$$\tilde{h}(f; \boldsymbol{\lambda}, \boldsymbol{\theta}) \propto v^{-7/2}\, e^{i\psi(v; \boldsymbol{\lambda}, \boldsymbol{\theta})}, \tag{7}$$

where $v^{-7/2}$ is the leading-order approximation to $A(v)$, and $\psi(v; \boldsymbol{\lambda}, \boldsymbol{\theta})$ is a post-Newtonian approximation to $\Psi(v)$. In Sec. 3, $\psi(v; \boldsymbol{\lambda}, \boldsymbol{\theta})$ will be given by the 2PN approximation obtained by Blanchet, Iyer, Will, and Wiseman [3] using post-Newtonian theory. In Sec. 4, $\psi(v; \boldsymbol{\lambda}, \boldsymbol{\theta})$ will be given by the slow-motion, perturbation-theory results of Tanaka, Tagoshi, and Sasaki [10]; this approximation does not incorporate the finite-mass-ratio terms of the true post-Newtonian series, but it is carried out to a high PN order.

It should be emphasized that the perturbation-theory results are valid only in the limit $\mu/M \to 0$. Nevertheless, I will still consider binary systems with large mass ratios, so as to give an indication of the robustness of my conclusions.

III. SECOND POST-NEWTONIAN WAVEFORMS AS SEARCH TEMPLATES

Serge Droz and I have calculated fitting factors for several binary systems using 2PN waveforms as templates. Our results are presented in the table, where for comparison I also display the fitting factors for Newtonian templates.

	advanced LIGO		initial LIGO
System	FF–Newton	FF–2PN	FF–2PN
$1.4 + 1.4\ M_\odot$	79.2%	93.0%	95.8%
$0.5 + 5.0\ M_\odot$	51.6%	95.2%	89.7%
$1.4 + 10\ M_\odot$	55.8%	91.4%	88.4%
$10 + 10\ M_\odot$	70.1%	91.7%	86.3%
$4 + 30\ M_\odot$	61.3%	86.6%	67.9%

We see that for the advanced version of LIGO and for most of the binary systems considered, the fitting factors are all above the 90% mark. According to Apostolatos' criterion [1], second post-Newtonian waveforms should make acceptably effective search templates. Additional details can be found in Ref. [7].

IV. PADÉ APPROXIMANTS

Recently, Damour, Iyer, and Sathyaprakash (DIS) [6] discovered a clever way of expressing the post-Newtonian waveforms in terms of Padé approximants, which accelerates the convergence of the post-Newtonian series. Here I shall restrict all considerations to the test-mass limit, $\mu/M \to 0$.

It is known that Padé approximants work especially well for rational functions. However, the orbital energy (per unit rest mass) of a test particle in Schwarzschild spacetime is not a rational function of the orbital velocity: $\tilde{E}(v) = (1 - 2v^2)/(1 - 3v^2)^{1/2}$. To circumvent this, DIS choose to employ the alternative energy function $\tilde{e}(v) \equiv \tilde{E}^2 - 1 = -v^2(1 - 4v^2)/(1 - 3v^2)$, which is rational. In a post-Newtonian context, only the first few

terms of a Taylor series about $v = 0$ can be calculated. For example, at the 2PN level, $\tilde{e}(v) \simeq -v^2(1 - v^2 + 3v^4)$. It is easy to check that the equivalent Padé approximant coincides with the *exact* result.

The Padé series therefore converges rapidly to the exact result for $\tilde{e}(v)$, and it may be used to estimate the position of the pole: $v_{\rm pole} = 1/\sqrt{3}$. In the test-mass limit, this estimate is exact. It is possible, however, that the estimate might not be as accurate in the finite-mass-ratio case, for which an exact expression is not available. As a way of challenging the DIS proposal, I will allow for a possible uncertainty in the position of the pole, by introducing a free parameter ξ such that $v_{\rm pole} = \xi/\sqrt{3}$.

The second element of the DIS proposal is to express $P(v)$, the luminosity function introduced in Sec. 2, also in terms of Padé approximants. This requires some thought, because schematically, the post-Newtonian series for $P(v)$ (as calculated by Tanaka, Tagoshi, and Sasaki [10]) takes the form

$$\begin{aligned} P(v) = 1 &+ v^2 + v^3 + v^4 + v^5 + (1 + \ln v)v^6 \\ &+ v^7 + (1 + \ln v)v^8 + (1 + \ln v)v^9 \\ &+ (1 + \ln v)v^{10} + (1 + \ln v)v^{11}; \end{aligned} \tag{8}$$

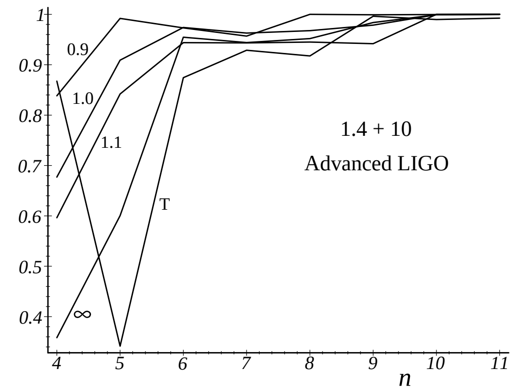
the presence of logarithmic terms prevents a straightforward conversion to a Padé form. The way around this, as implemented by DIS, is to factorize these terms. DIS also choose to factorize a simple pole at $v = v_{\rm pole}$, because this pole, already present in the energy function, can be shown to appear also in the luminosity function. The new expression for $P(v)$ is therefore

$$\begin{aligned} P(v) = &\big[1 + (v^6 + \cdots + v^{11}) \ln v\big] \\ &\times \big[1 - v/v_{\rm pole}\big]^{-1} \\ &\times \big[1 + v + \cdots + v^{11}\big], \end{aligned} \tag{9}$$

and DIS re-express the third factor in terms of its equivalent Padé approximant.

Using the DIS proposal, I have calculated semi-maximized ambiguity functions (SMAF) for selected binary systems. The results depend on the order n at which the post-Newtonian expansion is truncated. [For example, $n = 6$ corresponds to Eq. (9) with all terms of order v^7 and higher discarded; this is the 3PN-Padé approximation.] The results depend also on ξ, which parameterizes the possible uncertainty in the position of the pole. The values $\{1.0, 0.9, 1.1, \infty\}$ were selected; the choice $\xi = \infty$ corresponds to the absence of the second factor in Eq. (9) — the pole is not factorized.

The representative case of a $1.4+10\ M_\odot$ binary system is represented in the figure, which plots the SMAF as a function of the truncation order n, calculated using a noise curve appropriate for an advanced LIGO detector. The curve labelled "T" is obtained by using the usual Taylor expansion (8) for the luminosity function, together with a Taylor expansion for $Q(v)$. The curves labelled by numbers are obtained by using the Padé form of Eq. (9),



with ξ given by the corresponding number. The curve labelled "∞" is also obtained by using the Padé form of Eq. (9), but in the absence of the second factor. The exact expression for $Q(v)$ is used for all the Padé curves.

We see that the Padé curves converge to unity much more rapidly than the Taylor curve. This demonstrates the great success of the DIS proposal. Furthermore, this is true for *all* the Padé curves, whether or not they incorporate an uncertainty in $v_{\rm pole}$. I find this robustness of the DIS method quite remarkable. It is also interesting to note that the choice $\xi = 0.9$ produces the *largest* values for the SMAF.

These results suggest that 3PN-Padé waveforms will make a suitable choice of search templates. They may also be sufficiently accurate for the reliable estimation of source parameters. Additional details will be presented elsewhere [9].

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